

Disordered systems and the replica method in AdS/CFT

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ABSTRACT: We formulate a holographic description of effects of disorder in conformal field theories based on the replica method and the AdS/CFT correspondence. Starting with n copies of conformal field theories, randomness with a gaussian distribution is described by a deformation of double trace operators. After computing physical quantities, we take the $n \rightarrow 0$ limit at the final step. We compute correlation functions in the disordered systems by using the holographic replica method as well as the formulation in the conformal field theory. We find examples where disorder changes drastically the scaling of two point functions. The renormalization group flow of the effective central charge in our disordered systems is also discussed.

KEYWORDS: AdS-CFT Correspondence, Random Systems.

Contents

1. Introduction	1
2. Disordered systems and the replica method	3
3. Disordered systems via AdS/CFT	5
3.1 Double trace deformation in AdS/CFT	6
3.2 Holographic replica method	7
3.3 Examples of disorder systems in AdS/CFT	11
3.3.1 Random $\mathcal{N} = 4$ super Yang-Mills	11
3.3.2 Random $O(N)$ magnet	11
3.4 Comments on replica symmetry breaking	11
4. Field theory analysis in the planar limit	12
5. Conclusions and discussions	14
A. Order/disorder phase transition	16
B. Effective central charges in disordered systems	16

1. Introduction

Classical statistical or quantum many-body systems can be studied in the presence of spatial inhomogeneity. Indeed, it is quite rare that a perfectly clean system is realized in experiments, and hence having a good understanding of effects of randomness/impurities is to some extent necessary. Besides such a practical motivation, disorder by itself or combined effects of disorder and interactions can give rise to rich phenomena, which deserve studies in their own right. To name a few, complex behaviors in spin glass systems such as the Ising model with random bonds (random ferromagnetic interactions) or random magnetic field [1], or Anderson localization of electronic systems in the absence/presence of electron-electron interactions [2, 3] have been discussed.

When the amount of disorder is small or disorder is (marginally) irrelevant in the renormalization group (RG) sense, effects of disorder can be studied perturbatively around a clean critical point. On the other hand, randomness is not necessarily small and it can drive the system to a new type of disorder-dominated critical point, called random critical point. Examples include a multicritical point in the random bond Ising model [1, 4], the integer quantum Hall plateau transition [5], and a possibility of metal-insulator transitions in (2+1)-dimensional correlated electron systems [6]. These putative critical

points are beyond perturbative or mean-field treatments and understanding the nature of these random critical points has remained to this date as a major challenge in condensed matter physics.

It is the purpose of this paper to discuss quenched disordered systems in the framework of AdS/CFT correspondence [7]. Effects of disorder have been studied as a perturbation to weakly coupled and exactly solvable (conformal) field theories, in terms of the perturbation theory or perturbative RG, assuming the disorder strength is small (see, e.g., [8–12]). On the other hand, if we apply the AdS/CFT correspondence, it might be possible to solve strongly disordered problems because AdS/CFT is a strong/weak duality. In AdS/CFT setups, CFTs are typically non-abelian gauge theories [7] or the critical $O(N)$ vector model [13]. Even though their precise relations to real condensed matter systems are not clear at present, recently there have been several hopes and circumstantial evidences that the AdS/CFT can capture essential features of condensed matter systems, such as the electrical and thermal transport [14], the quantum Hall plateau transition [15], the superconductivity [16], the entanglement entropy [17], the scale invariant theories with non-trivial dynamical exponents [18], and so on. In the presence of weak randomness a holographic analysis for disordered systems was given in [19].

Assuming that disorder configurations such as the location of impurities are distributed according to some underlying probability distribution, quantities of our interest (e.g., local correlation functions) also fluctuate from different disorder configurations. We are thus forced to deal with the probability distribution of observables or in particular the first (or first few) moment(s) of the observables. At this point, it is important to emphasize that the average over disorder configurations is taken *after* we take the statistical mechanical ensemble average over spins, or the path integral over (quantum) field configurations and so on. This is a major difficulty in disordered systems since for a given distribution of disorder we do not have translation invariance, although correlation functions after quenched disorder averaging may be translation invariant. A standard tool to analyze effects of disorder is *the replica method (or replica trick)* (see, e.g., [20, 21, 10] and section 2 in the present paper).

In this paper, we consider a CFT deformed by a certain operator with its coupling changing from position to position. We will show that with the replica method, a generalization of the double trace interaction [22, 23] can nicely describe the randomness. Based on this idea, we will formulate the holographic replica method with some examples in section 3. In particular, we calculate the two point functions and show that their scaling behaviors importantly change due to the randomness. In section 4, we will give a complementary field theoretic analysis of the same system and confirm that it agrees with the holographic result. In appendix A, we will present a generalized holographic replica model and realize the order/disorder phase transition.

In contrast to clean systems, the structure of the RG flows connecting random critical points is less understood in disordered systems. This is so since critical theories describing random critical points are expected to be non-unitary. This in turn means we cannot use, in two dimensions, say, the c -theorem by Zamolodchikov, to know the direction of the RG flow. In particular, when the supersymmetric disorder averaging is applicable,

the central charge is always vanishing $c = 0$ because of the cancellation among matter fields and ghosts. Gurarie [24, 12] introduced, with the use of the supersymmetric disorder averaging, the effective central charge (or Gurarie's b) and proposed to use it to measure the degrees of freedom of a disordered two-dimensional CFT. Later, the equivalent quantity in the replica method (which we call c_{eff} in this paper) was introduced by Cardy [25]. (See appendix B.) It is an open problem if the effective central charge c_{eff} shows the irreversible relation $c_{\text{eff}}^{\text{IR}} < c_{\text{eff}}^{\text{UV}}$ along the RG flow, as the central charge in two-dimensional unitary CFTs does. In appendix B, we take a first step of the holographic calculation of c_{eff} and show that the c -theorem like relation $c_{\text{eff}}^{\text{IR}} < c_{\text{eff}}^{\text{UV}}$ holds in examples where we can calculate c_{eff} straightforwardly.

2. Disordered systems and the replica method

Let us start by reviewing how the replica method can be applied to quenched disordered systems (see, for a review, e.g. [20, 21, 10]).¹ Our arguments cover any quantum field theory (QFT) and we simply represent the action of a d -dimensional QFT as $S_0[\varphi] = \int d^d x \mathcal{L}_0(\varphi)$ with abstractly expressing all fields as φ . We pick up a certain operator in this theory and denote it by $\mathcal{O}(x)$. We perturb this theory by adding an interaction of the form

$$S = S_0 + \int d^d x g(x) \mathcal{O}(x). \tag{2.1}$$

This defines a classical disordered system in d spacial dimensions. In a $d + 1$ -dimensional quantum system, a spatially inhomogeneous perturbation by disorder is given by $\int dt d^d x g(x) \mathcal{O}(x, t)$, where t represents the (imaginary) time direction, and the disorder configuration $g(x)$ depends only on the spacial coordinates x . Below we shall proceed in the classical setup, though we can extend to the latter case straightforwardly.

A quenched disordered system (or random system) is such a system where the coupling $g(x)$ is depending randomly on the spatial coordinates x . We assume that the randomness is distributed with a gaussian profile, i.e., its distribution functional is given by

$$P[g(x)] \propto e^{-\frac{1}{2f} \int d^d x g(x)^2} \tag{2.2}$$

with $f > 0$. The free energy in the disordered system $\overline{\log Z}$ can be found by simply taking the gaussian average of $\log Z$ over disorder, where $\overline{\dots}$ represents disorder averaging with respect to the distribution $P[g(x)]$.

In disordered systems, we are interested in correlation functions averaged over disordered configurations. An important point is that this random average is not equivalent to taking the random average for the action itself (2.1). Instead the averaged correlation

¹There are two types of disorder: annealed disorder and quenched disorder. The former is the case where impurities are in thermal equilibrium with main system, while in the latter, it is not. In this paper we only discuss the latter case. Refer to [10] for more details.

function is expressed as follows;

$$\overline{\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_k)\rangle} = \int Dg(x)P[g(x)] \left[\frac{\int D\varphi e^{-S_0[\varphi]-\int d^d x g(x)\mathcal{O}(x)}\mathcal{O}(x_1)\mathcal{O}(x_2)\cdots\mathcal{O}(x_k)}{\int D\varphi e^{-S_0[\varphi]-\int d^d x g(x)\mathcal{O}(x)}} \right]. \tag{2.3}$$

The most important technical problem is how to deal with the factor of the inverse partition function. One way to overcome this difficulty is to introduce ghosts and represent the inverse partition function by that of ghosts. This is called the supersymmetric method and at a random fixed point it is typically described by a non-unitary CFT with the vanishing total central charge (see, e.g., [26, 12, 9]). This procedure is quite useful since we can utilize the knowledge of quantum field theory technique. However, it has a severe disadvantage that presently it can be used only for limited situations, where the original theory described by S_0 is a free field theory.

Instead of proceeding along this direction, we would like to resort to another method called the replica method, which can be applied to any quantum field theories (refer to, for example, [20]). The method may be summarized as follows. First we introduce n copies of the QFT and denote the i -th copy as QFT_i . Next we prepare n copies of the field φ_i in QFT_i . Then we consider the path integral in the product of n QFTs, i.e. $\text{QFT}_1 \otimes \text{QFT}_2 \otimes \cdots \otimes \text{QFT}_n$, given by

$$\int Dg(x)P[g(x)] \prod_{i=1}^n [D\varphi_i] e^{-\sum_{i=1}^n S_0[\varphi_i]-\int d^d x g(x)\sum_{i=1}^n \mathcal{O}_i(x)}\mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\cdots\mathcal{O}_1(x_k) \\ = \int \prod_{i=1}^n [D\varphi_i] e^{-\sum_{i=1}^n S_0[\varphi_i]+\frac{f}{2}\int d^d x (\sum_{i=1}^n \mathcal{O}_i(x))^2}\mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\cdots\mathcal{O}_1(x_k), \tag{2.4}$$

where $\mathcal{O}_1(x)$ is the operator $\mathcal{O}(x)$ in the Hilbert space of QFT_1 . In the replica method, we identify the average (2.3) with the $n \rightarrow 0$ limit of (2.4). The parameter n is originally a positive integer number, but we assume the analytical continuity with respect to n and take the limit $n \rightarrow 0$ finally. Indeed the inverse partition function required in (2.3) formally appears in this limit.

Let us define the scaling dimension of $\mathcal{O}(x)$ by Δ , then the deformation induced by the randomness is relevant or marginal if

$$2\Delta \leq d. \tag{2.5}$$

This condition is clear from power counting in the replicated action (2.4) and is called Harris criterion.

We consider large N gauge theories or a $O(N)$ vector model as examples of QFT since they appear in AdS/CFT correspondence. The operator \mathcal{O} that couples to disorder $g(x)$ is then given by a single trace operator, and with the replica trick the disorder effect is expressed by the double trace deformations as in (2.4), which have been studied in the context of AdS/CFT [22, 23, 27]. If we are interested in strongly coupled (or strongly

disordered) regime of field theories, then the path integral in (2.4) would be quite difficult to compute. Since the AdS/CFT correspondence maps strongly coupled CFTs to weakly coupled gravity theories, it is desirable to establish dual description to analyze strongly coupled disordered systems. In this paper we mainly focus on the planar limit of large N gauge theory, which is dual to the classical limit of the dual gravity theory.

3. Disordered systems via AdS/CFT

Now we move on to the main part of this paper: formulation of the AdS/CFT correspondence for disordered systems. We consider a standard setup of AdS/CFT, where a d dimensional Euclidean CFT is dual to a gravity theory on $d + 1$ dimensional Euclidean anti-de Sitter (AdS) space described by the metric

$$ds^2 = \frac{dz^2 + \sum_{\mu} dx^{\mu} dx_{\mu}}{z^2}. \tag{3.1}$$

The dual CFT is proposed to live on the boundary located at $z = 0$.

A spin-less operator \mathcal{O} , which is typically a single trace operator in a gauge theory, is supposed to be dual to a scalar field² ϕ with a mass m in the bulk AdS space. The relation between the conformal dimension Δ of \mathcal{O} and the mass m is given by the formula [28, 29]

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{m^2 + \frac{d^2}{4}}. \tag{3.2}$$

For our purpose, we need a relevant operator in the replicated theory, and hence we require (2.5). From this condition we pick up the smaller dimension Δ_- in (3.2) and we call it simply Δ below. The behavior of ϕ near the boundary $z = 0$ looks like

$$\phi(z, x) \sim z^{d-\Delta}(\alpha(x) + O(z^2)) + z^{\Delta} \left(\frac{\beta(x)}{2\Delta - d} + O(z^2) \right). \tag{3.3}$$

In the standard interpretation of AdS/CFT [31], $\alpha(x)$ is regarded as a source to the dual operator \mathcal{O} , while $\beta(x)$ is its expectation value $\langle \mathcal{O}(x) \rangle = \beta(x)$. We demand the normalizability for the mode with $\phi \sim z^{\Delta}$, which leads to a constraint on the range of Δ . Combining with (2.5) we have to choose³

$$\frac{d-2}{2} \leq \Delta \leq \frac{d}{2}. \tag{3.4}$$

The lower bound is known to be dual to the unitarity bound in the dual CFT [31].

For the scalar field ϕ we require the regularity at $z = \infty$, then $\alpha(x)$ and $\beta(x)$ are related as follows

$$\beta(x) = \pi^{-d/2} \frac{(2\Delta - d)\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \int d^d x' \frac{\alpha(x')}{|x - x'|^{2\Delta}}. \tag{3.5}$$

²We will stick to the scalar field example just for simplicity. We can generalize this to, say, vector and spinor fields straightforwardly.

³In order for Δ to satisfy this condition, the mass of ϕ should be in the range of $-d^2/4 < m^2 < (1 - d^2)/4$. In AdS space tachyonic modes are allowed due to the curvature, and the lower bound is known as Breitenlohner-Freedman bound [30].

In the momentum space representation it can be expressed as

$$\beta(k) = G(k)\alpha(k), \tag{3.6}$$

where $G(k)$ is given by⁴

$$G(k) = \frac{(2\Delta - d)\Gamma(d/2 - \Delta)}{\Gamma(\Delta - d/2)} \left(\frac{k}{2}\right)^{2\Delta-d}. \tag{3.8}$$

Since we chose Δ_- instead of Δ_+ as the conformal dimension of dual operator \mathcal{O} , the term proportional to $\alpha(x)$ is less singular than the one proportional to $\beta(x)$ in the boundary limit as opposed to the usual cases. Despite this fact we can still treat the former as a source because $\alpha(x)$ and $\beta(x)$ are related to each other via a canonical transformation as first claimed by Klebanov and Witten [31].

3.1 Double trace deformation in AdS/CFT

According to [23] (see also [27]) a multi-trace deformation $\int d^d x W(\mathcal{O})$ can be incorporated into the formulation of AdS/CFT correspondence. Here we focus on the deformation of a CFT by a double trace operator of the form ($\lambda > 0$)

$$S_{\text{int}} = \frac{\lambda}{2} \int d^d x (\mathcal{O}(x))^2, \tag{3.9}$$

as a preparation for the later analysis of disordered systems. Assigning the boundary behavior (3.3) and demanding the regularity at $z \rightarrow 0$, the scalar field is uniquely parameterized by $\alpha(x)$ or $\beta(x)$. As mentioned before these two parameters can be exchanged by a Legendre transform, and it is useful for our purpose to use $\beta(x)$ instead of $\alpha(x)$. Inserting the scalar field into the kinetic term and partially integrating over the coordinate z , we can obtain the action in terms of $\alpha(x)$. Then the Legendre transform leads to

$$S[\beta] = \frac{1}{2} \int d^d k \frac{\beta(k)\beta(-k)}{G(k)}. \tag{3.10}$$

Notice that the result is expressed as a field theory on the boundary of AdS space.

Recall that the expectation value of $\mathcal{O}(x)$ corresponds to the variable $\beta(x)$. Then the total action S in the presence of the double trace deformation (3.9) and a source to ϕ is expressed as

$$S[\beta, J] = \int d^d k \left[\frac{1}{2} \beta(k) \left(\frac{1}{G(k)} + \lambda \right) \beta(-k) + \beta(-k) J(k) \right]. \tag{3.11}$$

The equation of motion for β leads to

$$\frac{1 + \lambda G(k)}{G(k)} \cdot \beta(k) + J(k) = 0. \tag{3.12}$$

⁴We employed the formula

$$\int d^d x \frac{e^{ikx}}{x^{2\Delta}} = 2^{d-2\Delta} \pi^{\frac{d}{2}} \frac{\Gamma(d/2 - \Delta)}{\Gamma(\Delta)} k^{2\Delta-d}. \tag{3.7}$$

In other words, the boundary condition for ϕ is now changed into $\alpha + \lambda\beta + J = 0$. From (3.12) we can express

$$S[J] = -\frac{1}{2} \int d^d k J(k) \left(\frac{G(k)}{1 + \lambda G(k)} \right) J(-k), \quad (3.13)$$

and in the end we obtain the two point function

$$\frac{\delta^2}{\delta J(k) \delta J(-k)} e^{-S[J]} \Big|_{J=0} = \langle \mathcal{O}(k) \mathcal{O}(-k) \rangle = \frac{G(k)}{1 + \lambda G(k)} \quad (3.14)$$

taking the derivatives with respect to the source $J(k)$.

Due to the deformation of the (marginally) relevant operator (3.9), the conformal symmetry should be broken in the CFT side. Nevertheless, the vacuum solution in the AdS side remains trivial $\phi(x, z) = 0$ even in the presence of non-vanishing λ . Hence the background is still AdS_{d+1} , which implies the conformal symmetry of dual CFT. This is so since our calculation neglects backreaction in the gravity theory, which should be taken into account from one-loop order [32]. Notice that the two point function (3.14) breaks the conformal invariant.⁵ In the UV limit $k \rightarrow \infty$, it is approximated by

$$\langle \mathcal{O}(k) \mathcal{O}(-k) \rangle \sim G(k) \sim k^{2\Delta-d}, \quad (3.15)$$

while in the IR limit we find

$$\langle \mathcal{O}(k) \mathcal{O}(-k) \rangle \sim \frac{1}{\lambda} - \frac{1}{\lambda^2 G(k)} \sim \text{const.} + O(k^{d-2\Delta}). \quad (3.16)$$

This shows that under the RG flow the operator with the conformal dimension $\Delta (= \Delta_-)$ flows to the one with the conformal dimension $d - \Delta (= \Delta_+)$ [23, 32, 33]. In the IR limit conformal invariance is recovered and the operator dual to the scalar field ϕ has the conformal dimension Δ_+ . This does not cause any problems since the modes for both Δ_- and Δ_+ are normalizable due to our restriction (3.4).

3.2 Holographic replica method

Now we are prepared to present the formulation of AdS/CFT for disordered systems by employing the replica method.⁶ If we consider a CFT with the random interaction (2.1), it is described by a deformation of the product of n copies of the CFT, $\text{CFT}_1 \otimes \text{CFT}_2 \otimes \dots \otimes \text{CFT}_n$, in the replica method. Assuming the gaussian distribution of disorder (2.2), the theory is deformed by double trace operators as (2.4)

$$S_{\text{int}} = -\frac{f}{2} \int d^d x \left(\sum_{i=1}^n \mathcal{O}_i(x) \right)^2, \quad (3.17)$$

⁵At the planar limit generic correlation functions do not receive any contributions from the deformation of double trace operator. This fact can be understood by explicitly writing down the double line Feynman diagrams. The correlation functions involving \mathcal{O} are exceptions.

⁶A slightly different replica method has been applied to AdS/CFT in [17] to compute the entanglement entropy [34].

where we denote \mathcal{O}_i as the operator \mathcal{O} with its conformal dimension Δ in CFT $_i$. Compared with the standard double trace deformation (3.9), the sign of the coupling (3.17) is opposite (since f in (2.2) must be positive), a common feature of the replicated theory in disordered systems. Thus, one may worry that this interaction may cause an instability in this theory. If we wish, we can add a conventional double trace deformation (3.9) to the original theory together with the randomness (2.1). In the replica method, it means that we consider the product of n CFTs with the following generalized interaction

$$S_{\text{int}} = -\frac{f}{2} \int d^d x \left(\sum_{i=1}^n \mathcal{O}_i(x) \right)^2 + \frac{\lambda}{2} \int d^d x \sum_{i=1}^n (\mathcal{O}_i(x))^2. \quad (3.18)$$

Later, however, we will see that the limit $\lambda = 0$ of two point functions is well-defined, and the two point functions show more interesting scaling behaviors than the case with $\lambda > 0$.

In the dual AdS side, the spacetime is defined by n copies of an AdS space. They are disconnected in the bulk and attached to the same boundary R^d at $z = 0$. As seen above, the deformation by the double trace operator changes the boundary condition for the dual field, therefore after the deformation by (3.18) the fields in copies of an AdS space interact with each other through the boundary conditions. Interestingly, this setup with multiple AdS spaces has recently been discussed in [35, 36] from a different motivation. Originally we have n gravitons dual to n stress-energy tensors conserved independently. After we put the interaction (3.18), only a combination of stress-energy tensors is conserved and hence there exists only one dual massless graviton as argued in [35, 36]. In the AdS gravity the other $n - 1$ gravitons become massive due to the one-loop contribution, while in the dual CFT, the conformal dimension of other combinations of stress-energy tensors deviates from $\Delta = d$ to $d + \delta(n)$ due to the interaction (3.18) [25, 35, 36]. The standard bulk to boundary relation relates $\delta(n)$ to the mass of $n - 1$ gravitons as $M_g^2 = d\delta(n)$.

In the following, we calculate two point functions at the tree level of AdS gravity in this setup. We denote ϕ_i as the field corresponding to \mathcal{O}_i , and define $\alpha_i(x)$ and $\beta_i(x)$ from the asymptotic behaviors as in (3.3). Following the previous analysis, we obtain the deformed action as

$$S[\beta, J]_n = \int d^d k \left[\frac{\sum_{i=1}^n \beta_i(k) \beta_i(-k)}{2G(k)} - \frac{f}{2} \left(\sum_{i=1}^n \beta_i(k) \right) \cdot \left(\sum_{i=1}^n \beta_i(-k) \right) + \frac{\lambda}{2} \sum_{i=1}^n \beta_i(k) \beta_i(-k) + \beta_1(-k) J_1(k) + \beta_2(-k) J_2(k) \right]. \quad (3.19)$$

Here we included the source terms only for β_1 and β_2 without losing generality, by taking the symmetry into account. Notice that when $\lambda > 0$, this system (3.19) is stable in the $n \rightarrow 0$ limit.⁷ Taking the limit of $\lambda \rightarrow 0$ in the end, we define the theory with $\lambda = 0$.

⁷Indeed, if we redefine $\hat{\beta}_0 = \frac{1}{\sqrt{n}} \sum_i \beta_i$, $\hat{\beta}_i = \beta_i - \frac{1}{n} \sum_j \beta_j$, then the interaction terms are given by $\frac{1}{2}(\lambda - nf)\hat{\beta}_0^2 + \frac{\lambda}{2} \sum_i \hat{\beta}_i^2$. Therefore, any $\lambda > 0$ would be enough to stabilize the saddle points for small n .

From the equations of motion $\delta S/\delta\beta_i = 0$, we find

$$\begin{aligned}\beta_1 &= \frac{-fG(k)J_2(k) - (1 + (\lambda + (1-n)f)G(k))J_1(k)}{(1 + \lambda G(k))(1 + (\lambda - nf)G(k))}G(k), \\ \beta_2 &= \frac{-fG(k)J_1(k) - (1 + (\lambda + (1-n)f)G(k))J_2(k)}{(1 + \lambda G(k))(1 + (\lambda - nf)G(k))}G(k), \\ \beta_3 &= \dots = \beta_n = -\frac{fG(k)^2(J_1(k) + J_2(k))}{(1 + \lambda G(k))(1 + (\lambda - nf)G(k))}.\end{aligned}\tag{3.20}$$

From these equations we can express the action S as

$$\begin{aligned}S[J]_n &= -\frac{1}{2} \int d^d k \left(\frac{G(k) (1 + (\lambda + f(1-n))G(k)) (J_1(k)J_1(-k) + J_2(k)J_2(-k))}{(1 + \lambda G(k))(1 + (\lambda - nf)G(k))} \right. \\ &\quad \left. + \frac{2fG(k)^2 J_1(k)J_2(-k)}{(1 + \lambda G(k))(1 + (\lambda - nf)G(k))} \right).\end{aligned}\tag{3.21}$$

While n is a positive integer in our starting expression, in (3.21) we are free to regard n as a continuous valuable, which is a crucial assumption for the replica method.

Two point functions can be computed from (3.21) as

$$\langle \mathcal{O}_1(k)\mathcal{O}_1(-k) \rangle_n = \frac{G(k) (1 + (\lambda + f(1-n))G(k))}{(1 + \lambda G(k))(1 + (\lambda - nf)G(k))},\tag{3.22}$$

$$\langle \mathcal{O}_1(k)\mathcal{O}_2(-k) \rangle_n = \frac{fG(k)^2}{(1 + \lambda G(k))(1 + (\lambda - nf)G(k))}.\tag{3.23}$$

The subscript n implies that the two point functions are evaluated in the replicated theory with fixed n . As discussed in section 2, we take the limit $n \rightarrow 0$ and finally find in the disordered system as

$$\overline{\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle} = \langle \mathcal{O}_1(k)\mathcal{O}_1(-k) \rangle = \frac{(1 + (f + \lambda)G(k)) G(k)}{(1 + \lambda G(k))^2}.\tag{3.24}$$

We can also compute $\overline{\langle \mathcal{O}(k)\rangle\langle \mathcal{O}(-k)\rangle}$, which involves, when replicated, two distinct replicas,

$$\overline{\langle \mathcal{O}(k)\rangle\langle \mathcal{O}(-k)\rangle} = \langle \mathcal{O}_1(k)\mathcal{O}_2(-k) \rangle = \frac{fG(k)^2}{(1 + \lambda G(k))^2}.\tag{3.25}$$

Unlike $\overline{\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle}$, $\overline{\langle \mathcal{O}(k)\rangle\langle \mathcal{O}(-k)\rangle}$ is made non-zero solely because of disorder, and hence our analysis indeed describes a disordered phase. Correlation functions of this type have been used as an order parameter in spin glass theories [1].

When $\lambda > 0$, we can observe from the two point function (3.24) that the operator \mathcal{O} with conformal dimension Δ always flows into the one with $\Delta_+ = d - \Delta$ in the IR limit $k \rightarrow 0$. Therefore, the IR limit of the disordered system looks similar to the theory deformed by a single double trace operator (3.9).⁸ In the appendix A, we suggest that the random spin system, such as the random bond Ising model [4], is analogous to this

⁸Fixed points for more generic cases with $n = 2$ are analyzed in [36] and their result is consistent with ours.

case with $\lambda = f$, where the order/disorder phase transition occurs. In the appendix B, we analyze the effective central charge c_{eff} [25, 8] of this disordered CFT and calculate the difference between the effective central charge at the UV (trivial) fixed point ($c_{\text{eff}}^{\text{UV}}$) and at the IR fixed point ($c_{\text{eff}}^{\text{IR}}$). We find that the inequality $c_{\text{eff}}^{\text{IR}} - c_{\text{eff}}^{\text{UV}} \leq 0$ always holds for any values of $\lambda > 0$. This may support an analogue of c -theorem in the disordered system, though there has been no proof of the c -theorem for c_{eff} from the CFT side.

At the special point $\lambda = 0$, we can obtain a markedly different result. In the IR limit $k \rightarrow 0$, the operator \mathcal{O} with dimension Δ becomes an operator with dimension $2\Delta - d/2$, as seen from the behavior

$$\overline{\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle} \sim k^{4\Delta-2d}. \quad (3.26)$$

The constraint (3.4) leads to $d/2 - 2 \leq 2\Delta - d/2 \leq d/2$, and hence the lower bound of $2\Delta - d/2$ violates the unitarity bound. However, this may be fine since the disordered system is not a closed unitary system. The analysis of the effective central charge c_{eff} at $\lambda = 0$ seems to need a special care as discussed in the appendix B and we leave it as a future problem. Observe that the IR limit is not fully gapped, but rather the theory flows into another critical field theory. As it becomes clear from the RG analysis in the next section, the random fixed point corresponds to the infinite randomness limit ($f \rightarrow \infty$). This behavior could be comparable with random vector potential models of Anderson localization [37] or random quantum spin chains [38].

In principle, we can extend our computations to higher point functions by including interaction terms in (3.21). In the case of a cubic coupling, we can add to (3.19) a cubic term like

$$S_{\text{cubic}} = \int d^d k_1 d^d k_2 C(k_1, k_2) \beta(k_1) \beta(k_2) \beta(-k_1 - k_2), \quad (3.27)$$

where $C(k_1, k_2)$ is related to the three point function in momentum space for $f = 0$ via the standard rule [28, 29].

It is also possible to add more general multi-trace interactions $\int d^d x W(\mathcal{O}(x))$ in the boundary CFT. Correspondingly, in the holographic description, we need to add the potential term $\int d^d x W(\beta(x))$ to (3.19), which leads to the generalized action

$$S = K \int d^d x d^d y \frac{\sum_{i=1}^n \beta_i(x) \beta_i(y)}{|x-y|^{2(d-\Delta)}} + \int d^d x \left[-\frac{f}{2} \left(\sum_{i=1}^n \beta_i(x) \right)^2 + \sum_{i=1}^n W(\beta_i(x)) \right], \quad (3.28)$$

where K is a numerical constant.

Similarly, we can consider more general probability distributions for disorder other than the gaussian white-noise. After the theory is replicated, it leads to a deformation

$$(-1) \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \int d^d x_1 \cdots \int d^d x_p \overline{g(x_1) \cdots g(x_p)}^c \sum_{a_1, \dots, a_p} \mathcal{O}_{a_1}(x_1) \cdots \mathcal{O}_{a_p}(x_p), \quad (3.29)$$

where $\overline{g(x_1) \cdots g(x_p)}^c$ is the p -th cumulant of the probability distribution $P[g(x)]$.

3.3 Examples of disorder systems in AdS/CFT

3.3.1 Random $\mathcal{N} = 4$ super Yang-Mills

As the first example, we can consider the disordered system of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions by a random deformation (2.1). We can take \mathcal{O} to be a 1/2 BPS operator with $\Delta = 2$ made of two transverse scalars, i.e. $\mathcal{O}_{ab} = \text{Tr}[\Phi^a \Phi^b] - \frac{1}{6} \delta^{ab} \text{Tr}[\Phi^c \Phi^c]$. In this case, the condition (2.5) is saturated and therefore the RG flow is logarithmic.

3.3.2 Random $O(N)$ magnet

Klebanov and Polyakov [13] conjectured that the massless fields in AdS_4 with even spins describe the singlet sector of the three-dimensional critical $O(N)$ vector model in the large N limit. By using the vector field $\vec{\phi}$ with N components, the action of the $O(N)$ vector model is

$$S = \frac{1}{2} \int d^3x \left[\partial \vec{\phi} \cdot \partial \vec{\phi} + \frac{\lambda}{2N} (\vec{\phi} \cdot \vec{\phi})^2 \right]. \quad (3.30)$$

There are two critical points; One is at $\lambda = 0$, i.e., the free field theory. The other is at the end point of the RG flow induced by the second term in (3.30), which is interpreted as a double trace deformation by setting $\mathcal{O} = \vec{\phi} \cdot \vec{\phi}$ in section 3.1. Notice that the dimension of $\mathcal{O}^2 = (\vec{\phi} \cdot \vec{\phi})^2$ is $2\Delta = 2$ and thus it is relevant. In the IR fixed point, the conformal dimension of \mathcal{O} changes into $d - \Delta = 2$ and then \mathcal{O}^2 becomes irrelevant.

Now, starting from the $O(N)$ vector model, we introduce the randomness via the interaction $\int d^3x g(x) \vec{\phi}(x) \cdot \vec{\phi}(x)$. In the replica method, this disordered system is described by the $n \rightarrow 0$ limit of the following system

$$S = \frac{1}{2} \int d^3x \left[\sum_{i=1}^n \partial \vec{\phi}_i \cdot \partial \vec{\phi}_i - \frac{f}{N} \sum_{i,j=1}^n (\vec{\phi}_i \cdot \vec{\phi}_i) (\vec{\phi}_j \cdot \vec{\phi}_j) + \frac{\lambda}{2N} \sum_{i=1}^n (\vec{\phi}_i \cdot \vec{\phi}_i)^2 \right]. \quad (3.31)$$

Its holographic description is precisely given by the model (3.19) analyzed in the previous subsection. We then conclude that whenever $\lambda \neq 0$, disorder is innocuous in the IR limit. In particular, disorder is an irrelevant perturbation at the non-trivial fixed point of (3.30), which agrees with the well-known fact for the 3D $O(N)$ magnets with $N \geq 2$. (See, for example, [39, 40].)

3.4 Comments on replica symmetry breaking

The replica symmetry is the symmetry under a permutation of fields among different replicas, such as $\beta_i \rightarrow \beta_j$ ($i \neq j$) in our setup. The previous analysis (3.19) clearly preserves the replica symmetry as all of β_i are vanishing. The breaking of the replica symmetry typically occurs when there are many vacua in the replica theory. If this happens, the analysis becomes more non-trivial because the definition of order parameter (or mean field) of randomness gets complicated. A famous such example is the problem of the spin-glasses (see e.g. [1]).

Indeed, we can find examples where the replica symmetry is spontaneously broken in our holographic setup as follows. If we are interested in the IR limit, we can drop off the

first term in (3.19) and it is straightforward to study the vacuum structure of this system from the potential terms. By choosing $W(\beta)$ appropriately,⁹ we can realize situations where the potential $\sum_i W(\beta_i)$ has only the trivial vacua,¹⁰ i.e., $\beta_i = 0$, while the potential with the term proportional to f has multiple vacua with $\beta_i \neq \beta_j$.

4. Field theory analysis in the planar limit

In this section, we show that the two point functions (3.24) and (3.25) obtained from our holographic method can be reproduced from field theoretic calculations in the planar limit. This confirms the validity of our formulation of the holographic replica method. Before considering the replicated case, we reproduce from the field theory calculations the two point function (3.14) in the case of the deformation of (3.9)

$$S_{\text{int}} = \lambda \int d^d x \Phi_{\text{pert}}(x), \quad \Phi_{\text{pert}}(x) = \frac{1}{2}(\mathcal{O}(x))^2. \quad (4.1)$$

In order to consider the renormalization of λ and \mathcal{O} , we need to compute beta function β_λ and anomalous dimension $\gamma_{\mathcal{O}}$. In the large N limit, non-trivial contributions come only from the two point function $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle = v/|x|^{2\Delta}$ with $v = \pi^{-d/2} \frac{(2\Delta-d)\Gamma(\Delta)}{\Gamma(\Delta-d/2)}$, and those from higher point functions are suppressed (see, e.g., [23]). Therefore, divergent terms arise only from the following operator product expansions (OPEs) as

$$\Phi_{\text{pert}}(x)\Phi_{\text{pert}}(0) \sim \frac{2v}{|x|^{2\Delta}}\Phi_{\text{pert}}(0), \quad \Phi_{\text{pert}}(x)\mathcal{O}(0) \sim \frac{v}{|x|^{2\Delta}}\mathcal{O}(0). \quad (4.2)$$

From these OPEs we can obtain the beta functions following a standard analysis of quantum field theories¹¹

$$\frac{d}{d \ln |k|} \tilde{\lambda}(k) = \beta_{\tilde{\lambda}} = (2\Delta - d)\tilde{\lambda}(k) + (\tilde{\lambda}(k))^2, \quad \gamma_{\mathcal{O}} = \Delta + \frac{1}{2}\tilde{\lambda}(k), \quad (4.3)$$

where we redefine $\tilde{\lambda} = \frac{2^{d-2\Delta}(2\Delta-d)\Gamma(d/2-\Delta)}{\Gamma(\Delta-d/2)}\lambda$. Notice that even though we included only the leading order corrections to λ , the result is exact in the large N limit. The beta function (4.3) leads to

$$\tilde{\lambda}(k) = \frac{(d-2\Delta)\tilde{\lambda}_0}{|k|^{d-2\Delta} + \tilde{\lambda}_0}. \quad (4.4)$$

Since the RG flow equation is given by

$$\left[\frac{d}{d \ln |k|} - \beta_\lambda \frac{d}{d\lambda} + d - 2\gamma_{\mathcal{O}} \right] \langle \mathcal{O}(k)\mathcal{O}(-k) \rangle = 0, \quad (4.5)$$

⁹For example, we can realize this situation if we assume $\frac{dW(\beta)}{d\beta}$ is an oscillating function so that its average is increasing and $\frac{dW(\beta)}{d\beta} = 0$ has only a single solution $\beta = 0$.

¹⁰Here we include metastable vacua into our definition of vacua.

¹¹For an extensive analysis of RG flows in the $n = 2$ case, refer to the third paper in [36].

the two point function is

$$\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle = \mathcal{C} \exp \left(- \int^{\ln|k|} d \ln |k'| (d - 2\gamma_{\mathcal{O}}(k')) \right) = \frac{G(k)}{1 + \lambda_0 G(k)}. \quad (4.6)$$

The coefficient \mathcal{C} is fixed such that $\langle \mathcal{O}(k)\mathcal{O}(-k) \rangle = G(k)$ for $\lambda_0 = 0$.

Next let us introduce furthermore disorder (2.1) with the gaussian distribution (2.2). In the replica method the disordered system can be represented by introducing n copies of the CFT with $\mathcal{O}_i(x)$ ($i = 1, 2, \dots, n$) whose OPEs are $\mathcal{O}_i(x)\mathcal{O}_j(0) \sim \delta_{i,j}v/|x|^{2\Delta}$. Moreover we deform the n copies of the CFT by (3.18)

$$S_{\text{int}} = -f \int d^d x \Phi_{\text{pert}}(x) + \lambda \int d^d x \Psi_{\text{pert}}(x), \quad (4.7)$$

$$\Phi_{\text{pert}}(x) = \frac{1}{2} \left(\sum_{i=1}^n \mathcal{O}_i(x) \right)^2, \quad \Psi_{\text{pert}}(x) = \frac{1}{2} \sum_{i=1}^n (\mathcal{O}_i(x))^2. \quad (4.8)$$

As in the previous analysis, we now reproduce the two point functions (3.24) and (3.25). From the OPE coefficients, we read off the beta functions as¹²

$$\begin{aligned} \frac{d}{d \ln |k|} \tilde{f}(k) &= \beta_{\tilde{f}} = (2\Delta - d)\tilde{f}(k) - n(\tilde{f}(k))^2 + 2\tilde{f}(k)\tilde{\lambda}(k), \\ \frac{d}{d \ln |k|} \tilde{\lambda}(k) &= \beta_{\tilde{\lambda}} = (2\Delta - d)\tilde{\lambda}(k) + (\tilde{\lambda}(k))^2. \end{aligned} \quad (4.9)$$

We find, therefore,

$$\tilde{f}(k) - \frac{\tilde{\lambda}(k)}{n} = \frac{(d - 2\Delta)(\tilde{f}_0 - \tilde{\lambda}_0/n)}{|k|^{d-2\Delta} + \tilde{\lambda}_0 - n\tilde{f}_0}, \quad \tilde{\lambda}(k) = \frac{(d - 2\Delta)\tilde{\lambda}_0}{|k|^{d-2\Delta} + \tilde{\lambda}_0}. \quad (4.10)$$

For computing the anomalous dimensions it is convenient to rotate the operators as

$$\hat{\mathcal{O}}_0(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathcal{O}_i(x), \quad \hat{\mathcal{O}}_j(x) = \mathcal{O}_j(x) - \frac{1}{n} \sum_{i=1}^n \mathcal{O}_i(x), \quad (4.11)$$

with $j = 1, \dots, n$, according to the irreducible representation of symmetric group \mathcal{S}_n . Here the number of independent operators does not change since $\sum_{j=1}^n \hat{\mathcal{O}}_j = 0$. In this normalization we have $\hat{\mathcal{O}}_0(x)\hat{\mathcal{O}}_0(0) \sim v/|x|^{2\Delta}$ and $\hat{\mathcal{O}}_i(x)\hat{\mathcal{O}}_j(0) \sim (\delta_{i,j} - \frac{1}{n})v/|x|^{2\Delta}$. The anomalous dimensions of new operators are obtained just like before as

$$\gamma_{\hat{\mathcal{O}}_0} = \Delta + \frac{1}{2}\tilde{\lambda} - \frac{n}{2}\tilde{f}, \quad \gamma_{\hat{\mathcal{O}}_j} = \Delta + \frac{1}{2}\tilde{\lambda}, \quad (4.12)$$

and solving the RG flow equation we have

$$\langle \hat{\mathcal{O}}_0(k)\hat{\mathcal{O}}_0(-k) \rangle_n = \frac{G(k)}{1 + (\lambda_0 - n f_0)G(k)}, \quad \langle \hat{\mathcal{O}}_i(k)\hat{\mathcal{O}}_j(-k) \rangle_n = \left(\delta_{i,j} - \frac{1}{n} \right) \frac{G(k)}{1 + \lambda_0 G(k)}. \quad (4.13)$$

¹²When specialized to the case of the random $O(N)$ magnet discussed in section 3.3.2, the beta functions are consistent with the known results to the leading order in N . See, for example, [39, 40].

Rotating the operators back again, now we can reproduce the previous results of the two point functions (3.22) and (3.23), therefore after taking $n \rightarrow 0$ limit we have (3.24) and (3.25).

The method with the RG flow equation might be a standard way to compute correlation functions, but there is another way in a field theory viewpoint via a Hubbard-Stratonovich transformation first considered by Gubser and Klebanov [33]. The partition function we consider can be rewritten as

$$Z_f^n[J] = \left(\det -\frac{1}{f}\right)^{\frac{1}{2}} \left(\det \frac{1}{\lambda}\right)^{\frac{n}{2}} \int Dg(x) \prod_{i=1}^n D\sigma_i(x) \times \left\langle e^{\int d^d x \left[-\frac{1}{2f}g(x)^2 + \frac{1}{2\lambda} \sum_{i=1}^n \sigma_i^2 + \sum_{i=1}^n (g(x) + \sigma_i(x) + J_i(x)) \mathcal{O}_i(x)\right]} \right\rangle_0, \quad (4.14)$$

where the subscript 0 suggests that the correlation function is computed without perturbation, i.e. with $f = \lambda = 0$. Integration over σ_i reproduces the double trace deformation $\frac{\lambda}{2} \int d^d x \sum_i \mathcal{O}_i^2$. Notice that the non-trivial contribution arises only from the two point function $\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle = \delta_{i,j} G(x) (\equiv \delta_{i,j} v/|x|^{2\Delta})$ in the large N limit. From this observation we find

$$Z_f^n[J] = \left(\det -\frac{1}{f}\right)^{\frac{1}{2}} \left(\det \frac{1}{\lambda}\right)^{\frac{n}{2}} \int Dg(x) \prod_{i=1}^n D\sigma_i(x) \times e^{\int d^d x \left[-\frac{1}{2f}g(x)^2 + \frac{1}{2\lambda} \sum_{i=1}^n \sigma_i^2(x) + \frac{1}{2} \sum_{i=1}^n (g(x) + \sigma_i(x) + J_i(x)) \hat{G}(g(x) + \sigma_i(x) + J_i(x))\right]}, \quad (4.15)$$

where $\hat{G}(g) = \int d^d y G(x-y)g(y)$. Since this expression is gaussian with respect to g and σ_i , we can easily integrate them out. The result is¹³

$$Z_f^n[J] = (1 + \lambda \hat{G})^{-\frac{n-1}{2}} (1 + (\lambda - nf) \hat{G})^{-\frac{1}{2}} \times e^{\frac{1}{2} \int d^d x \left[\sum_{i=1}^n J_i(x) \hat{Q} J_i(x) + (\sum_{i=1}^n J_i(x)) \hat{Q}' (\sum_{i=1}^n J_i(x))\right]}, \quad (4.16)$$

with

$$\hat{Q} = \frac{\hat{G}}{1 + \lambda \hat{G}}, \quad \hat{Q}' = \frac{f \hat{Q}^2}{1 - nf \hat{Q}}. \quad (4.17)$$

Taking derivatives with respect to J_i twice, we obtain the two point functions (3.22) and (3.23) and again reproduce (3.24) and (3.25) in the $n \rightarrow 0$ limit.

5. Conclusions and discussions

In this paper, we have studied the quenched disordered systems with arbitrary strength of disorder by applying the AdS/CFT correspondence. We formulate the holographic replica method by employing the setup of the double trace deformation in AdS/CFT. We have calculated the two point functions in our disordered system in the planar limit from both

¹³The prefactor corresponds to the $O(N^0)$ corrections to the partition function, and it can be used to compute the shift of central charge along the RG flow. See, for example, [33] and appendix B.

the AdS and CFT sides and got the same result. We found that the scaling of the two point functions evolves non-trivially under the RG flow. Especially, if we fine tune the parameter to $\lambda = 0$, then the two point functions in the IR limit show remarkably new behavior. In the generic case $\lambda > 0$, the IR limit is essentially the same as the fixed point obtained by the standard double trace deformation. As in appendix B it is possible to analyze the effective central charge c_{eff} using the AdS/CFT correspondence. There we observe that it decreases under the RG flow between two fixed points when $\lambda > 0$. A more thorough analysis of this would be very intriguing as no proof of the c -theorem about c_{eff} has been known so far.

It is important to remember that the subtle limit $n \rightarrow 0$ of the replica method does not cause any problems in our examples. We also pointed out that the replica symmetry may be broken if we consider particular deformations of the CFT. It may also be intriguing to note that the limit $n \rightarrow 0$ offers us a formal way to construct AdS duals to non-unitary CFTs with the central charge $c = 0$.

Even though our holographic formalism of the replica method covers quantum disordered systems, our explicit computations are performed only for classical disordered systems. Thus here we would like to mention the application of our holographic replica method to quantum disordered systems. Since the random coupling $g(x)$ does not depend on the time t , the frequency ω of the field β_i in (3.19) should be vanishing. Thus the results (3.24) and (3.25) remain the same only when $\omega = 0$; Otherwise we will get the same result as the one in the pure system, i.e., with $f = 0$. This triviality is because we are taking the planar limit in the presence of the multi-trace interactions. In order to obtain ω -dependent results, we need to go beyond the tree level analysis in the gravity theory. A similar situation occurs when we are considering two point functions of other operators, such as the currents $J^\mu(t, x)$. At the tree level, we do not find any f -dependence in both classical and quantum disorder systems. These problems of one loop analysis clearly deserve future studies.

In addition to the replica method, we know another method which also enables us to study disordered systems, called the supersymmetric method, as mentioned before. In order to deal with the randomness in a theory with scalar fields and Dirac fermions we add their superpartners (or ghosts). In this procedure, the bosonic global symmetry, such as $O(N)$, becomes its supergroup extension, such as $OSp(N|N)$. It may be interesting to find a similar method for Yang-Mills gauge theories. Naively, one may think that we can replace the bosonic gauge group $U(N)$ with the supergroup $U(N|M)$ (see [41] for gauge theories with supergroup gauge symmetries). However, we can easily see that this leads to what we precisely want only at the vanishing coupling $g_{\text{YM}} = 0$. Therefore we need a modification or another idea.

Acknowledgments

We would like to thank T. Azeyanagi, M. Fukuma, S. Hartnoll, C. Herzog, S. Iso, J. McGreevy and T. Nishioka for useful conversations. We are also grateful to E. Kiritsis for a useful comment. The work of YH is supported by JSPS Research Fellowship. SR thanks

Center for Condensed Matter Theory at University of California, Berkeley for its support. The work of TT is supported by JSPS Grant-in-Aid for Scientific Research No.18840027 and by JSPS Grant-in-Aid for Creative Scientific Research No. 19GS0219.

A. Order/disorder phase transition

In condensed matter systems, randomness often competes with ordering tendencies. For example, for a spin system (e.g. the Ising model) with critical temperature T_c , the system is in an ordered phase (e.g. ferromagnetic phase) for $T < T_c$. Even if we introduce a small amount of disorder the system will still be in the ordered phase. However, if the randomness becomes strong enough, a phase transition will occur and the system will evolve into a disordered phase. This is a standard story, for example, in the ferromagnets in the presence of random magnetic field (see e.g. [10]).

Such competition can be described in our holographic approach. In the spin system examples, we regard \mathcal{O} as the spin operator σ . Then an important point is that we omit the $i = j$ terms in the interaction terms $-\frac{f}{2} \sum_{i,j} \mathcal{O}_i \mathcal{O}_j$ (see (3.18)) since the operator $:\mathcal{O}^2:$ does not exist in spin systems like the Ising model.¹⁴ Another point is that when $T < T_c$ and $f = 0$, the system is in a ordered phase. As an example we express this with the spontaneous breaking of the Z_2 symmetry $\beta \rightarrow -\beta$ by adding the standard wine bottle potential $W(\beta) = -m^2\beta^2 + \lambda\beta^4$. Notice that $m^2 > 0$ when $T < T_c$, while $m^2 = 0$ at $T = T_c$. In summary, we reach the following action of β

$$S[\beta]_n = \int d^d k \left[\frac{\sum_{i=1}^n \beta_i^2}{2G} - \frac{f}{2} \sum_{i \neq j}^n \beta_i \beta_j + \sum_i (-m^2 \beta_i^2 + \lambda \beta_i^4) \right]. \tag{A.1}$$

Assuming that the vacuum preserves the replica symmetry, i.e., $\beta_1 = \beta_2 = \dots = \beta_n \equiv \beta$, we can rewrite (A.1) as follows

$$S[\beta]_n = n \int d^d k \left[\frac{1}{2G} \beta^2 - \frac{f}{2} (n-1) \beta^2 - m^2 \beta^2 + \lambda \beta^4 \right]. \tag{A.2}$$

In the $n \rightarrow 0$ limit, the term due to the randomness behaves like $\frac{f}{2} \beta^2$ and it competes with the spontaneous breaking of the Z_2 symmetry. Therefore we can realize the disorder/order phase transition when we increase the randomness parameter f in our holographic description.

B. Effective central charges in disordered systems

We can measure the degrees of freedom of a given CFT in even dimensions by calculating the central charge. Let us suppose that the replica theory has a non-trivial fixed point.

¹⁴For example, if we consider the Ising model, the OPE of the spin operators is $\sigma \cdot \sigma = I + \mathcal{E}$. This OPE produces the energy density operator \mathcal{E} , but this just shifts the critical temperature. In contrast, if we consider the large N $O(N)$ vector model, then the operator $:\mathcal{O}^2:$ exists and we have to keep it as in (3.19). Even in the latter case, we can add the interaction term as in (3.18) with $\lambda = f$ and realize a system without the $i = j$ terms.

Then we can define the central charge of this new CFT as

$$c(n) = nc_0 + \Delta c(n), \quad (\text{B.1})$$

for any fixed n . Here c_0 is the central charge of the original CFT without any deformations. For example, in the $\mathcal{N} = 4$ $U(N)$ gauge theory case it is given by $c_0 = N^2/4$. The term $\Delta c(n)$ is due to the deformations and comes from one-loop order corrections $O(N^0)$. In the random system, we can still define so called the effective central charge c_{eff} (see e.g. [25, 8])

$$c_{\text{eff}} \equiv \left. \frac{dc(n)}{dn} \right|_{n=0}. \quad (\text{B.2})$$

Physically, this central charge is equal to the coefficient of the OPE of energy stress tensors at the random fixed point [25]

$$\overline{\langle T_{\mu\nu} T_{\mu'\nu'} \rangle} \propto c_{\text{eff}}. \quad (\text{B.3})$$

In a usual unitary CFT, there is a c -theorem, which states that the central charge is decreasing under the RG flow. However, in disordered systems, no c -theorem is known for c_{eff} even if the replicated theory with fixed n is unitary, because finally we need to take the formal $n \rightarrow 0$ procedure.

In our setup of AdS/CFT, the order $O(N^0)$ correction to the central charge is proportional to the one from the one-loop effective potential $V_{1\text{-loop}} = -\frac{1}{2}\text{Tr}_{\text{AdS}}(-\square + m^2)$ of the scalar field. Since our formulation of holographic replica model is closely related to the case with the double trace deformation (3.9), let us first review the case analyzed in [32]. In the case with the double trace deformation (3.9), the λ -dependence of V_{double} comes from the boundary condition for the field $\phi(x, z)$ imposed at the boundary $z = 0$. For $\alpha(x)$ and $\beta(x)$ in (3.3), we assign

$$\alpha(x) = -\lambda\beta(x), \quad (\text{B.4})$$

which can be obtained from the relations (3.12) and (3.6). The one-loop vacuum energy was explicitly calculated in [32] as

$$V_{\text{double}}(\lambda) = -\frac{1}{2^{d-2}\pi^{d/2}\Gamma(d/2)R^{d+1}} \int_0^\nu d\tilde{\nu} \frac{\tilde{\nu}}{\Gamma(\tilde{\nu})\Gamma(1-\tilde{\nu})} \int_0^\infty dp \frac{p^{d-1}\tilde{\lambda}}{p^{2\tilde{\nu}} + \tilde{\lambda}} K_{\tilde{\nu}}(p)^2, \quad (\text{B.5})$$

where $\nu = \frac{d}{2} - \Delta$ and $\tilde{\lambda} = 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \lambda z^{2\nu}$. Here z is the radial coordinate and R is the radius of AdS_{d+1} . In the above expression, we subtracted the divergent piece which does not depend on f , and which would actually be canceled by other contributions due to the supersymmetry. From this result we can show that $V_{\text{double}}(\lambda = \infty) < 0$, and this is consistent with the c -theorem as it is proportional to $c_{\text{IR}} - c_{\text{UV}}$.

Now we compute the one-loop vacuum energy for the product of n copies of the CFT with the deformation (3.18). In this case the boundary conditions are modified as

$$\alpha_j = f \sum_{i=1}^n \beta_i - \lambda\beta_j \quad (\text{B.6})$$

for $j = 1, 2, \dots, n$, or performing the redefinitions as in footnote 7,

$$\hat{\alpha}_0 = -(\lambda - nf)\hat{\beta}_0, \quad \hat{\alpha}_j = -\lambda\hat{\beta}_j. \quad (\text{B.7})$$

Here we should notice that $\sum_j \hat{\alpha}_j = \sum_j \hat{\beta}_j = 0$ by construction. Therefore, by comparing with the previous result, we find

$$V_{\text{replica}}(f, \lambda, n) = V_{\text{double}}(\lambda - nf) + (n - 1)V_{\text{double}}(\lambda). \quad (\text{B.8})$$

Notice that the identity $V_{\text{replica}}(f, \lambda, n = 0) = 0$ holds as expected. What we are interested in is the shift of the effective central charge c_{eff} (B.2) under the RG flow;¹⁵

$$\Delta c_{\text{eff}} = \left. \frac{d\Delta c(n)}{dn} \right|_{n=0} = A (V_{\text{double}}(\lambda) - f \cdot V'_{\text{double}}(\lambda)), \quad (\text{B.9})$$

where A is a positive constant.¹⁶

We would like to calculate the difference between c_{eff} at the UV fixed point $\lambda = f = 0$ and at the IR fixed point. If we assume $\lambda > 0$, then c_{eff} of the IR fixed point can be obtained by setting $\lambda = \infty$ (recall the analysis of the RG flow in section 4). Then we immediately find

$$c_{\text{eff}}^{\text{IR}} - c_{\text{eff}}^{\text{UV}} = A \cdot V_{\text{double}}(\infty) < 0. \quad (\text{B.10})$$

This seems to support the c -theorem like property for the effective central charge of our disordered CFT. However, we would like to stress again that the c -theorem for c_{eff} has not been proven at present and a counter example may be found in more generic cases. This value of $c_{\text{eff}}^{\text{IR}} - c_{\text{eff}}^{\text{UV}}$ (B.10) is actually the same as that of the standard double trace deformation [32, 33] even in the presence of the random perturbation.

Moreover, if we restrict ourselves to the case with $0 < f < \lambda$, then we can show that Δc_{eff} is always non-positive for any values of f and λ from the expression (B.5). Therefore we conclude that Δc_{eff} is a monotonically decreasing function of f and λ in this case.

In the remaining special case $\lambda = 0$, we would get naively the opposite result $c_{\text{eff}}^{\text{IR}} - c_{\text{eff}}^{\text{UV}} > 0$. However, this analysis is not reliable because $V_{\text{double}}(-nf)$ in (B.8) is divergent for any values of $n > 0$. This subtlety arises since at $\lambda = 0$ the system of (3.19) becomes unstable as mentioned before. We may instead deform the theory by, for example, a quartic term β^4 so that it become stabilized. We would like to leave the analysis of c_{eff} at $\lambda = 0$ as a future problem.

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¹⁵Strictly speaking, the notion of central charge can be used only at the conformal points. Therefore we should regard (B.9) as a definition of an analogue of Zamolodchikov's c -function.

¹⁶It is explicitly given by $A = c_0 \frac{8\pi G_N^{(5)} R^2}{2d}$.

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